1 Approximation

1.1 Newton-Raphson process

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

1.2 Linear interpolation

Draw triangles, use similar triangles.

1.3 Interval bisection

a	$f(a)$	b	$f(b)$	$\frac{a+b}{2}$	$f\left(\frac{a+b}{2}\right)$
2	-1	3	2	2.5	0.1569
2	-1	2.5	0.1569	2.25	-0.493

2 Summation of Series

2.1 Summation of Series

$$\sum_{x=1}^{n} x = \frac{n(n+1)}{2}$$

$$\sum_{x=1}^{n} x^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{x=1}^{n} x^3 = \frac{n^2(n+1)^2}{4}$$

2.2 Summation of Arithmetic Progression

$$S_n = a_1 n + \frac{(n)(n-1)d}{2}$$

$$S_n = a_0 n + \frac{(n)(n+1)d}{2}$$

$$S_n = \frac{n \times (a_1 + a_n)}{2}$$

$$S_n = n \times a_{\frac{n+1}{2}}$$

2.3 Summation of Geometric Progression

$$S_n = \frac{a_1 \times (1 - q^n)}{1 - q}$$

$$S_\infty = \frac{a_1}{1 - q}$$

3 Matrices

3.1 Transformations

3.1.1 Enlargement

- Stretch in x-direction by a scale factor k: $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$
- Stretch in y-direction by a scale factor k: $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$
- Enlargement about the origin by a scale factor k: $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

3.1.2 Reflection

- Reflection in x-axis: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- Reflection in y-axis: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
- Reflection in $y = x$: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- Reflection in $y = -x$: $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

3.1.3 Rotation

- Rotation about the origin by θ anti-clockwise: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

3.1.4 A point to a point

Point A is transformed by T, resultant point is $T \cdot A$.

3.1.5 A line to a line

Equation $\begin{pmatrix} a_1 + tb_1 \\ a_2 + tb_2 \\ a_3 + tb_3 \end{pmatrix}$ is transformed by T, resultant line is $T \cdot \begin{pmatrix} a_1 + tb_1 \\ a_2 + tb_2 \\ a_3 + tb_3 \end{pmatrix}$.

3.2 Inverse matrix 2*2

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\det A = ad - bc$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

If $\det A = 0$, A is singular, so A has no inverse.

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

3.3 Inverse matrix 3*3

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} = \frac{1}{\Delta} \begin{pmatrix} A & -B & C \\ -D & E & -F \\ G & -H & I \end{pmatrix}^{T}$$

where

$$A = \begin{vmatrix} e & f \\ h & i \end{vmatrix} = ei - hf$$

$$\Delta = aA - bB + cC$$

3.3.1 Transpose

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{T} = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

That is, rows become corresponding columns.

3.4 Calculating area of an triangle

$\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$

$$A = \frac{1}{2}\left(x_2y_1 + x_3y_2 + x_1y_3 - x_1y_2 - x_2y_3 - x_3y_1\right)$$

4 Complex Numbers

1) Translation

   $w = z + a + bi$: translation by $\begin{pmatrix} a \\ b \end{pmatrix}$

2) Enlargement

   $w = kz$: enlargement by a scale factor k

3) Enlargement followed by translation

   $w = kz + a + bi$: enlargement by a scale factor k followed by a translation by $\begin{pmatrix} a \\ b \end{pmatrix}$

4.1 Transformations

4.1.1 Example 1

Find the transformation $w = \frac{1}{z}, z! = 0$, find the locus of w when z lies on the line with equation $y = 2x + 1$

$$x + yi = \frac{1}{u + vi} = \frac{u - vi}{u^2 + v^2} = \frac{u}{u^2 + v^2} + \frac{-v}{u^2 + v^2}i$$

4.2 Eigenvectors

To find eigenvectors of A:

$$\det(A - \lambda I) = 0$$

Given an eigenvector e, find corresponding eigenvalue:

$$A{\cdot}e = \lambda e$$

Orthogonal matrix M:

$$M^-1 = M^T$$

5 Vector

5.1 Scalar product (·)

$$\vec{a} \cdot \vec{b} = x_1x_2 + y_1y_2 + z_1 + z_2$$

5.1.1 Angle between $\vec{a}$ and $\vec{b}$

$$\cos\theta = \frac{\vec{a}\vec{b}}{|\vec{a}|\left|\vec{b}\right|}$$

$$\vec{a} \cdot \vec{b} = |\vec{a}|\left|\vec{b}\right|\cos\theta$$

5.2 Vector product (×)

Vector product is perpendicular to both the vectors.

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

$$\left|\vec{a} \times \vec{b}\right| = |\vec{a}| \left|\vec{b}\right| \sin\theta$$

$$\vec{a} \times \vec{b} = |\vec{a}| \left|\vec{b}\right| \sin\theta \cdot \vec{u}$$

5.3 Calculate Area of a Triangle

$$A = \frac{1}{2} |\vec{a}| \left|\vec{b}\right| \sin\theta = \frac{1}{2} \left|\vec{a} \times \vec{b}\right|$$

5.4 Calculate Area of a Tetrahedron

$$A = \frac{1}{6}\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) \sin\theta$$

5.5 Calculate Area of a Prism

$$A = \frac{1}{6}\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) \sin\theta$$

5.6 Cartesian equation of a straight line

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3} = \lambda$$

where

$$fixedpoint(a_1, a_2, a_2)$$

,

$$direction = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

5.7 Vector equation of a straight line

$$\vec{r} \times \vec{b} = \vec{a} \times \vec{b}$$

$$\vec{r} = \vec{a} + t\vec{b}$$

5.8 Find the distance from a point to the line using vector product

$$\frac{\left|\vec{d}\right|}{\left|\vec{AP}\right|} = \sin\theta$$

$$\vec{d} = \frac{\left|\vec{AP} \times \vec{b}\right|}{\left|\vec{b}\right|}$$

5.9 Find the shortest distance between two lines

- Two parallel lines, choose one point from a line and calculate by

$$\vec{d} = \frac{\left|\vec{AP} \times \vec{b}\right|}{\left|\vec{b}\right|}$$

.

- Two skew lines:

$$\vec{r_1} = \vec{a} + \lambda\vec{b} \quad (1)$$

$$\vec{r_2} = \vec{c} + \lambda\vec{d} \quad (2)$$

$$d = \left|\frac{(\vec{a} - \vec{c}) \cdot \left(\vec{b} \times \vec{d}\right)}{\left|\vec{b} \times \vec{d}\right|}\right|$$

5.10 Vector Equation of a Plane

$$\vec{n} \cdot (\vec{r} - \vec{a}) = 0$$

5.11 Cartesian Equation of a Plane

$$ax + by + cz = d = \vec{n} \cdot \vec{a}, \vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

where $\vec{a}$ is a fixed point.

5.12 Transform a Plane from Cartesian Equation to Vector Equation

let $x = \lambda, y = \mu$, express z with x and y. $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then replace x with λ, split λ and μ.

5.13 Determine a Plane with two lines

1. Find normal of the plane by $\vec{b_1} \times \vec{b_2}$
2. Find intersection
3. Express in cartesian form

5.14 Find the distance between a point and a plane

Plane: $\vec{r} \cdot \vec{n} = ax + by + cz = d$, Point: $P(x_1, y_1, z_1)$

5.14.1 Method 1, use formulae directly

$$d = \frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

5.14.2 Method 2, use perpendicular foot

F is the perpendicular foot of P to the plane.

Step 1. Find for line PF, which is a expression of F

Step 2. F is in the plane, so put F into the equation of the plane and find F.

Step 3. Calculate distance by $\left|\vec{PF}\right|$

5.15 Line and plane

5.15.1 The line lies in the plane

Method 1. Prove two random points lie in the plane

Method 2. Prove all points are in the plane

5.15.2 The line is parallel to the plane

Method 1. $\vec{b}\vec{n} = 0$,

5.15.3 The line is intersecting with the plane

Method 1. Find point of intersection

Method 2. Find angle between l and the plane

5.16 Tow planes

5.16.1 Distance between two planes

$$ax + by + cz = d_1 \tag{3}$$

$$ax + by + cz = d_2 \tag{4}$$

$$d = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

6 Differentiation

6.1 First order differentiation

$$f(x)\frac{dy}{dx} + f'(x)y = \frac{d(f(x)y)}{dx}$$

Integration factor: $\boxed{e^{\int p dx}}$

$$\frac{dy}{dx} + py = Q \Rightarrow \frac{d(\boxed{e^{\int p dx}}y)}{dx} = \boxed{e^{\int p dx}}Q$$

6.2 Second order differentiation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

6.2.1 Auxiliary equation

$$am^2 + bm + c = 0$$

If $\Delta > 0$, it has two distinct roots α, β. General solution:

$$y = Ae^{\alpha x} + Be^{\beta x}$$

If $\Delta = 0$, it has two repeated roots. General solution:

$$y = (A + Bx)e^{\alpha x}$$

If $\Delta < 0$, it has two complex roots, $p + qi$ and $p - qi$. General solution:

$$y = e^{px}(A\cos qx + B\sin qx)$$

6.2.2 Example for finding a general solution for Second order differentiation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$
$$a = 1, b = 5, c = 6$$
$$m^2 + 5m + 6 = 0$$
$$m = -2 or m = -3$$
$$y = Ae^{-2x} + Be^{-3x}$$

6.2.3 Complementary functions

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

Solution: $y = complementary\ function + particular\ integral$

Particular integral is the general form of $f(x)$.

6.2.4 Complementary functions example

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 36x$$

Step 1. State CF and PI
CF: $y = Ae^{2x} + Be^{6x}$
PI: $y = \lambda x + \mu$

Step 2. Differentiate PI
Obtain:

$$\frac{dy}{dx} = \lambda$$
$$\frac{d^2y}{dx^2} = 0$$

Step 3. Substitute $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$, y into the differentiation equation.
Then find λ and μ.

6.3 Appendix: Particular Integrals

$f(x)$	Particular integral
k	λ
$ax + b$	$\lambda x + \mu$
$ax^2 + bx + c$	$\lambda x^2 + \mu x + \gamma$
ae^{kx}	λe^{kx}
$a \sin kx$ $a \sin kx$ $a \sin kx + b \cos kx$	$\lambda \sin kx + \mu \cos kx$

7 Maclaurin and Taylor series

7.1 Maclaurin expansion

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^r(0)}{r!}x^r + \ldots$$

7.1.1 Provided expansions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, -1 < x < 1$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots, -1 < x < 1$$

7.2 Taylor expansion

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^r(a)}{r!}(x-a)^r + \dots$$

$$f(x+a) = f(a) + f'(a)x + \frac{f''(a)}{2!}x^2 + \frac{f^r(a)}{r!}x^r + \dots$$

8 Polar Coordinates

8.1 Sketching Graphs in Polar Coordinates

8.2 Integration in Polar Coordinates

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$$

8.3 Differentiation in Polar Coordinates

Polar function $r = f(\theta)$ can be transformed to

$$y = r\sin\theta$$

$$x = r\cos\theta$$

Then differentiation:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

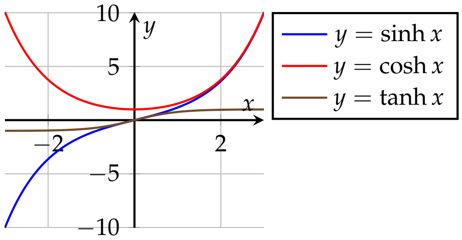
- For tangent parallel to initial line, $\frac{dy}{dx} = 0$, hence $\frac{dy}{d\theta} = 0$.
- For tangent perpendicular to initial line, $\frac{dy}{dx}$ is undefined, hence $\frac{dx}{d\theta} = 0$

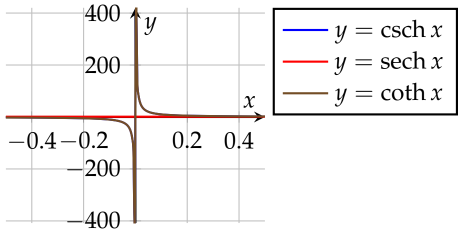
9 Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

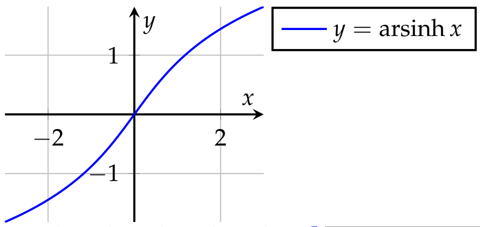


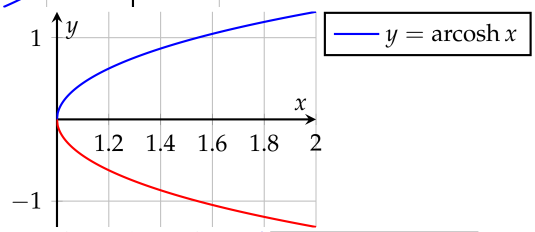


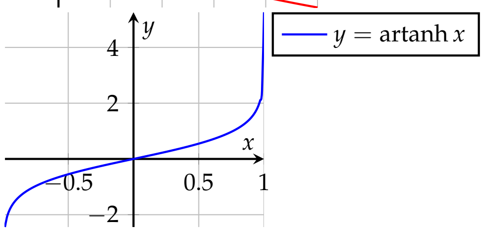
$$\operatorname{arsinh} x = \ln{(x + \sqrt{x^2+1})}$$

$$\operatorname{arcosh} x = \ln{(x \pm \sqrt{x^2-1})}$$

$$\operatorname{artanh} x = \frac{1}{2}\ln{(\frac{1+x}{1-x})}$$







9.1 Osborn's rule

Replace sin with sinh, cos with cosh, $\sin^2$ with $-\sinh^2$

10 Further integration

10.1 General formulae

$$\int f'(x) f^n(x) dx = \frac{f^{n+1}(x)}{n+1}$$

10.2 Useful formulae

$$\cosh 2x = 2\cosh^2 x - 1 = 1 + 2\cosh^2(x)$$
$$\cosh^2(x) = \frac{\cosh 2x + 1}{2}$$
$$\sinh^2(x) = \frac{\cosh 2x - 1}{2}$$

$$\frac{d}{dx}(\operatorname{arsinh} x) = \frac{1}{\sqrt{1+x^2}}$$
$$\frac{d}{dx}(\operatorname{arcosh} x) = \frac{1}{\sqrt{x^2-1}}$$
$$\frac{d}{dx}(\operatorname{artanh} x) = 1 - x^2$$
$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$
$$\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$$
$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$
$$\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin(\frac{x}{a}) + C$$

(Results marked (*) are in the Edexcel formula booklet)

$$\int \sinh x \, dx = \cosh x + C \text{ (*)}$$
$$\int \cosh x \, dx = \sinh x + C \text{ (*)}$$
$$\int \operatorname{sech}^2 x \, dx = \tanh x + C$$
$$\int \operatorname{cosech}^2 x \, dx = -\coth x + C$$
$$\int \operatorname{sech} x \tanh x dx = -\operatorname{sech} x + C$$
$$\int \operatorname{cosech} x \coth x dx = -\operatorname{cosech} x + C$$
$$\int \frac{1}{\sqrt{(a^2 - x^2)}} dx = \arcsin\left(\frac{x}{a}\right), \ |x| < a \quad (*) \qquad \int \frac{1}{\sqrt{(1 - x^2)}} dx = \arcsin x + C, \ |x| < 1$$
$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) \quad (*) \qquad \int \frac{1}{1 + x^2} dx = \arctan x + C$$
$$\int \frac{1}{\sqrt{(a + x^2)}} dx = \operatorname{arsinh}\left(\frac{x}{a}\right) \quad (*) \qquad \int \frac{1}{\sqrt{(1 + x^2)}} dx = \operatorname{arsinh} x + C$$
$$\int \frac{1}{\sqrt{(x^2 - a^2)}} dx = \operatorname{arcosh}\left(\frac{x}{a}\right), \ x > a \quad (*) \qquad \int \frac{1}{\sqrt{(x^2 - 1)}} dx = \operatorname{arcosh} x + C$$
$$\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a}\ln\left|\frac{a + x}{a - x}\right|, \ |x| < a \quad (*)$$
$$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\ln\left|\frac{x - a}{x + a}\right| \quad (*)$$

10.3 Substitution in Integration

$\int f(x)dx$	Substitution
$\int \frac{1}{\sqrt{a^2-x^2}}dx$ $\int \sqrt{a^2 - x^2}dx$	$x = a\sin\theta$
$\int \frac{1}{\sqrt{x^2-a^2}}dx$ $\int \sqrt{x^2 - a^2}dx$	$x = a\cosh\theta$
$\int \frac{1}{\sqrt{x^2+a^2}}dx$ $\int \sqrt{x^2 + a^2}dx$	$x = a\sinh\theta$
$\int \frac{1}{x^2+a^2}dx$	$x = a\tan\theta$

10.4 Arc Length

$$S = \int_{x_A}^{x_B} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$S = \int_{y_A}^{y_B} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$S = \int_{t_A}^{t_B} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

10.5 Surface Area

Rotating about x-axis

$$S = 2\pi \int_{x_A}^{x_B} y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$S = 2\pi \int_{x_A}^{x_B} y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dx$$

Rotating about y-axis

$$S = 2\pi \int_{x_A}^{x_B} x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$S = 2\pi \int_{x_A}^{x_B} x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dx$$

11 Further coordinates

Conics

	Ellipse	Parabola	Hyperbola	Rectangular Hyperbola
Standard Form	$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$	$y^2 = 4ax$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$xy = c^2$
Parametric Form	$(a\cos\theta, b\sin\theta)$	$(at^2, 2at)$	$(a\sec\theta, b\tan\theta)$ $(\pm a\cosh\theta, b\sinh\theta)$	$\left(ct, \frac{c}{t}\right)$
Eccentricity	$e < 1$ $b^2 = a^2(1 - e^2)$	$e = 1$	$e > 1$ $b^2 = a^2(e^2 - 1)$	$e = \sqrt{2}$
Foci	$(\pm ae, 0)$	$(a, 0)$	$(\pm ae, 0)$	$(\pm\sqrt{2}c, \pm\sqrt{2}c)$
Directrices	$x = \pm\frac{a}{e}$	$x = -a$	$x = \pm\frac{a}{e}$	$x + y = \pm\sqrt{2}c$
Asymptotes	none	none	$\frac{x}{a} = \pm\frac{y}{b}$	$x = 0, y = 0$

11.1 Ellipses

11.1.1 Gradient of tangent for ellipse

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\cos\theta}{-a\sin\theta}$$

11.2 Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

11.2.1 Asymptotes

$$y = \pm \frac{b}{a} x$$

11.2.2 Intersections

$$x = \pm a$$

11.2.3 Parametric equations

$$\begin{cases} x = a \sec \theta \\ y = b \tan \theta \end{cases}$$

$$\begin{cases} x = a \cosh \theta \\ y = b \sinh \theta \end{cases}$$

11.2.4 Differentiation

$$\frac{dy}{dx} = \frac{b}{a} \csc \theta$$

$$\frac{dy}{dx} = \frac{b}{a} \coth \theta$$

11.3 Eccentricity

$$e = \frac{distance\ to\ focus}{distance\ to\ directrix}$$

- If $0 < e < 1$, it's an ellipse. $foci(\pm ae, 0)$. directrix: $x = \pm \frac{a}{e}$
- If $e = 1$, it's an parabola.

  Eccentricity for ellipse:

$$b^2 = a^2(1 - e^2)$$

$$e^2 = 1 - \frac{b^2}{a^2}$$

  Eccentricity for hyperbola:

$$a^2 = b^2(e^2 - 1)$$

$$e^2 = 1 + \frac{a^2}{b^2}$$

12 Appendix: Formulas of Integration and Differentiation

$$\int \frac{f'(x)}{f(x)} dx = \ln(f(x))$$

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

$$\int \sec x dx = \ln(\sec x + \tan x) + C$$

$$\int \csc x dx = -\ln(\csc x + \cot x) + C$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$